Biometry 726
Fall 2010
Homework 2
Due Tuesday 5th October 2010

1. 3.11
2. 4.3
3. 4.4
4. We showed in class that for $\mathbf{X}_p \sim \mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $\mathbf{A}$ a $q \times p$ matrix of constants, and $\mathbf{b}$ a $q \times 1$ vector of constants,
$$\mathbf{Y} = \mathbf{A}\mathbf{x} + \mathbf{b} \sim \mathrm{N}_q(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}').$$
Use this result directly to show that if $\mathbf{X} = (X_1 \quad X_2 \quad X_3)'$ has density $\mathrm{N}_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\boldsymbol{\mu} = (\mu_1 \quad \mu_2 \quad \mu_3)'$ and
$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix},$$
then $\mathbf{X}_1 = (X_1 \quad X_2)' \sim \mathrm{N}_2(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$ where $\boldsymbol{\mu}_1 = (\mu_1 \quad \mu_2)'$ and
$$\boldsymbol{\Sigma}_1 = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}.$$
Hint: Consider the general proof provided on Page 158, and choose an appropriate matrix $\mathbf{A}$ *and vector* $\mathbf{b}$ *to apply the above theorem.*
5. Let $\mathbf{X} = (X_1 \quad X_2 \quad X_3)' \sim \mathrm{N}_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\boldsymbol{\mu} = \mathbf{0}$ and
$$\boldsymbol{\Sigma} = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$
Find the covariance of Z_1 and Z_2 where
$$\begin{aligned} Z_1 &= X_1 - X_2 + 2X_3 - 6 \quad \text{and} \\ Z_2 &= 2X_2 + 4. \end{aligned}$$
6. The moment generating function, $m_{\mathbf{X}}(\mathbf{t})$, of a 3×1 random vector $\mathbf{X} = (X_1 \quad X_2 \quad X_3)'$ is given by
$$m_{\mathbf{X}}(\mathbf{t}) = \exp\left\{t_1 - t_2 + 2t_3 + t_1^2 + \frac{1}{2}t_2^2 + 2t_3^2 - \frac{1}{2}t_1t_2 - t_1t_3\right\}.$$
Find a constant k such that $\mathrm{Prob}[2X_1 - 3X_2 + X_3 > k] = 0.95$.
7. Let $\mathbf{Y}_p \sim \mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.
   (a) Use the theorem shown in Question 4 (above) to show that
   $$\mathbf{Z} = \boldsymbol{\Sigma}^{-1/2}(\mathbf{Y} - \boldsymbol{\mu}) \sim \mathrm{N}_p(\mathbf{0}, \mathbf{I})$$
   where $\boldsymbol{\Sigma}^{-1/2}$ is obtained using the spectral decomposition, and $\mathbf{I}$ is the p-dimensional identity matrix.

(b) What is the marginal distribution of Z_i, the ith component of $\mathbf{Z}$? Justify your answer.

(c) Write a function in `R` called `my.rmvnorm` that returns n random draws from a multivariate normal distribution using a spectral decomposition approach, and based on the relationship between the components of $\mathbf{Z}$ and the distribution of $\mathbf{Y}$. Specifically, a call to your function should look like

```
my.rmvnorm(n,mu,sigma).
```

You may not use the existing `R` function, `rmvnorm`, to draw your sample.

(d) Demonstrate your sampler works by drawing a sample of 10,000 observations from a 3-dimensional multivariate normal distribution with mean and variance-covariance as described in 5, and calculating the proportion of observations for which

$$(\mathbf{Y} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \boldsymbol{\mu})$$

is less than or equal to $\chi^2_3(0.1)$ and $\chi^2_3(0.05)$.