Outline

• Examples using AMOS

• Identification
  • a) review

• Three useful types of SEMs
  – MIMIC model
  – multitrait-multimethod SEM
  – causal indicators

• Group comparison models
  – two-way interactions across two groups
  – two-way interactions across more than two groups
  – three-way interactions across two groups
  – three-way interactions across more than two groups
Rules for Variances

- \( \text{Var}(A + B) = \text{Var}(A) + \text{Var}(B) + 2\text{Cov}(A,B) \)
- \( \text{Var}(A - B) = \text{Var}(A) + \text{Var}(B) - 2\text{Cov}(A,B) \)

Let \( c \) be any constant (like, 5)
- \( \text{Var}(c + A) = 0 + \text{Var}(A) \) (constants don’t vary)
- \( \text{Var}(cA) = c^2\text{Var}(A) \)

Rules for Covariances

- \( \text{Cov}(X,Y) = E(XY) - E(X)E(Y) \)
- \( \text{Cov}(c,X) = 0 \)
- \( \text{Cov}(cX,Y) = c\text{Cov}(X,Y) \)
- \( \text{Cov}(X+Y,Z) = \text{Cov}(X,Z) + \text{Cov}(Y,Z) \)
Equations for path diagram:

\[ x_1 = \lambda_{11} \xi_1 + \delta_1 \]
\[ x_2 = \lambda_{21} \xi_1 + \delta_2 \]
\[ x_3 = \lambda_{31} \xi_1 + \delta_3 \]

Writing out equations:
\[
\text{Var}(x_1) = \text{Cov}(\lambda_{11} \xi_1 + \delta_1, \lambda_{11} \xi_1 + \delta_1) \quad \text{distribute (FOIL)}
\]
\[ = \text{Cov}(\lambda_{11} \xi_1, \lambda_{11} \xi_1) + \text{Cov}(\lambda_{11} \xi_1, \delta_1) + \text{Cov}(\delta_1, \lambda_{11} \xi_1) + \text{Cov}(\delta_1, \delta_1) \]
\[ = \text{Var}(\lambda_{11} \xi_1) + 2\text{Cov}(\lambda_{11} \xi_1, \delta_1) + \text{Var}(\delta_1) \]

* \( \text{Var}(cX) = c^2(X) \), \text{ also, assume that } \delta \text{ is uncorrelated with } X

\[ = \lambda_{11}^2 \text{Var}(\xi_1) + 0 + \text{Var}(\delta_1) \]
Similarly,
\[ \text{Var}(x_2) = \lambda_{21}^2 \text{Var}(\xi_1) + \text{Var}(\delta_2) \]
\[ \text{Var}(x_3) = \lambda_{31}^2 \text{Var}(\xi_1) + \text{Var}(\delta_3) \]

\[ \text{Cov}(x_1, x_2) = \text{Cov}(\lambda_{11} \xi_1 + \delta_1, \lambda_{21} \xi_1 + \delta_2) \]
\[ = \text{Cov}(\lambda_{11} \xi_1, \lambda_{21} \xi_1) + \text{Cov}(\lambda_{21} \xi_1, \delta_2) + \text{Cov}(\delta_1, \lambda_{21} \xi_1) + \text{Cov}(\delta_1, \delta_2) \]

\text{No covariance between } \xi_s, \delta_s \text{ and } \delta_s \text{ are uncorrelated}

\[ = \lambda_{11} \lambda_{21} \text{Cov}(\xi_1, \xi_1) + 0 + 0 = \lambda_{21} \lambda_{11} \text{Var}(\xi_1) \]

Similarly,
\[ \text{Cov}(x_3, x_1) = \lambda_{31} \lambda_{11} \text{Var}(\xi_1) \]
\[ \text{Cov}(x_3, x_2) = \lambda_{31} \lambda_{21} \text{Var}(\xi_1) \]

Why do all these calculations?

- Facilitates deeper understanding
- Highlights model assumptions
- Aids in determining if model is identified
Is model identified?

- 3 indicator rule?
  - ✓ at least one factor
  - ✓ at least 3 indicators
  - ✓ each x pointed to by only one $\xi$
  - ✓ $\delta$s are uncorrelated

- But, on a deeper level, why is a 3 indicator model identified?

How many things are we estimating?

- vars of exog. vars: 1
- vars of errors for endog: 3
- direct effects: 3
- double-headed arrows: 0 +
- Total to be Estimated: 7

$\lambda$s: (direct effects) we know:

\[
\text{Cov}(x_2, x_1) = \lambda_{21}\lambda_{11} \text{Var} (\xi_1)
\]

\[
\text{Cov}(x_3, x_1) = \lambda_{31}\lambda_{11} \text{Var} (\xi_1)
\]

\[
\text{Cov}(x_3, x_2) = \lambda_{31}\lambda_{21} \text{Var} (\xi_1)
\]

4 unknowns, 3 equations
Variances of $\delta$s

We know:

\[
\delta_1 = x_1 - \lambda_{11} \xi_1 \\
\delta_2 = x_2 - \lambda_{21} \xi_1 \\
\delta_3 = x_3 - \lambda_{31} \xi_1
\]

\[
\text{Var}(\delta_1) = \text{Var}(x_1) - \lambda_{11}^2 \text{Var}(\xi_1) \\
\text{Var}(\delta_2) = \text{Var}(x_2) - \lambda_{21}^2 \text{Var}(\xi_1) \\
\text{Var}(\delta_3) = \text{Var}(x_3) - \lambda_{31}^2 \text{Var}(\xi_1)
\]

Variance of $\xi_1$:

We know:

\[
\text{Var}(\xi_1) = \lambda_{11}^2 \text{Var}(\delta_1) + \text{Var}(\delta_1) \\
\text{Var}(\xi_1) = \lambda_{21}^2 \text{Var}(\delta_2) + \text{Var}(\delta_2)
\]

Verdict: as things stand, we don’t have enough information to estimate all the parameters. Per t-rule: $3(4)/2 = 6$

Solution: Fix one of the parameters values to 1

Set $\text{Var}(\xi_1)$ to 1, or a $\lambda$ e.g., $\lambda_{11}$ to 1

Unstandardized estimates

Set $\text{Var}(\xi_1)$ to 1

Set $\lambda_{11}$ to 1
standardized estimates

Set Var(ξ₁) to 1

Set λ₁ to 1

It doesn’t matter what you fix (with standardized estimates).

General identification rules for confirmatory factor analysis:

1) t-rule

2) Three indicator rule. Model passes rule if:
   a) it has three or more indicators per latent variable
   b) each row of Λₓ with one and only one nonzero element
   c) a diagonal Θ₉

3) Two indicator rule. Model passes rule if:
   a) factor complexity of each x is one
   b) no zero elements in Φ
   c) Θ₉ is diagonal
General identification rules for models that incorporate both structure equations and measurement models.

a) t-rule

b) two-step rule
   1) step 1:

   2) step 2:

If both the measurement model and the structural model are identifiable then the model as a whole is identifiable.

Two-Step Rule examples

\[ \eta = \gamma_1 \xi + \zeta \]

\[
\begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix} = 
\begin{bmatrix}
    \lambda_1 \\
    \lambda_2
\end{bmatrix} \xi + 
\begin{bmatrix}
    \delta_1 \\
    \delta_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
    y_1 \\
    y_2
\end{bmatrix} = 
\begin{bmatrix}
    \lambda_3 \\
    \lambda_4
\end{bmatrix} \eta + 
\begin{bmatrix}
    \epsilon_1 \\
    \epsilon_2
\end{bmatrix} \]
Step 1: Measurement Model (has observed variables), doesn’t assume anything about relationship between latent variables:

\[ \begin{align*}
\xi & \quad \eta \\
\delta_1 & \quad \delta_2 \\
x_1 & \quad x_2 \\
y_1 & \quad y_2 \\
\varepsilon_1 & \quad \varepsilon_2
\end{align*} \]

OK per 2-indicator rule

Step 2: Structural Model consists of the relationships between latent constructs – think of the latent variables as observed (in that their variances are known)

\[ \begin{align*}
\xi & \quad \eta
\end{align*} \]

OK per Null B rule

Bollen, pg 326-9

TWO STEP RULE EXAMPLES:
STEP 1: CFA

OK per 3-indicator rule

STEP 2: Structural Part

Null B-rule:
Recursive Rule:

T-Rule:
vars of exog. vars: 0
vars of errors for endog: 3
direct effects: 4
double-headed arrows: 0 +

Sample Moments = (3*4)/2=6
MIMIC (Multiple Indicators, Multiple Causes)

Three major components to the model: causes (left) indicators (right)
Latent construct (middle).

Quality of Life Research, 7, pp. 387-397

The proxy problem: child report versus parent report in health-related quality of life research

N. C. M. Theunissen*, T. G. C. Vogels, H. M. Koopman,
G. H. W. Verrips, K. A. H. Zwinderman, S. P. Verloove-Vanhorick
and J. M. Wtr
Identification of multi-trait multi-method SEMs

Must have at least three traits and three methods.

(Alwin, 1973)
Multiple-Group Comparisons (Interaction Effects)
1) Used to study interaction effects
2) SEMs well-suited to study interaction effects because they take
measurement error into account
   a) example: suppose you had a four-indicator measure of self-esteem and
   you hypothesized that it had a different relationship with depression for
   high SES and low SES adults [a dichotomous variable]. How would you
   go about testing this hypothesis with regular regression analysis?
      1) first, you would have to \textit{a priori} add the self-esteem indicators
      2) second, you would have to use product terms, which compound
   problems of reliability.
      a) we know that the $V_{(t1)} = \rho_1 V_{(p1)}$ and $V_{(t2)} = \rho_2 V_{(p2)}$
      b) so, then, the true variance of $V1$ times $V2$ (if they are not
       correlated) = $\rho_1 \rho_2 V_{(p1)} V_{(p2)}$
   b) recap:
      1) first, with an SEM it is possible to measure latent variables
      2) second, with an SEM it is possible to perform group comparisons
      and avoid the compounding of measurement error that is inherent
      in product terms.
Multiple-Group Comparisons: An Interaction Across Two Groups

1) Does gender modify the relationship between personableness and the democratic candidate’s success in winning a debate?
   a) check to make sure that the measurement models do not differ between groups.

   b) calculate model fit using a multiple group solution with no constraints across groups (except for measurement model)

   c) calculate model fit using a multiple group solution with an across-group constraint imposed to reflect the interaction effect.

   d) calculate the difference in model fit by comparing the fit index for the constrained solution with the fit index for the unconstrained solution.

Example: Ratings of a political debate

Number of parameters to estimate = 17
Straight arrows: 9 variances: 7
Curved arrows: 1
Multiple group comparison: Men and Women

Group 1: women

Group 2: men

Number of parameters to estimate: (17*2 = 34)

Multiple group comparison:

Saturated model: No constraints in parameters

Default model: All model parameters differ across groups (males and females)

Model #2: Measurement model, in which cfas are constrained to be equal across groups (males and females)
   a) lp1f=lp1m; lp2f=lp2m;
      lq1f=lq1m; lq2f=lq2m;
      ls1f=ls1m; ls2f=ls2m; ls3f=ls3m

Model #3: Same as Model #2, except that parameter of interest is constrained to be equal across groups
   a) additional constraint: ponsmale=ponsfem
Question 1: Measurement Model

Before looking to see if the relationship between “personableness” and “success” is the same for males and females, we first need to make sure that the latent constructs of “personableness” and “success” are the same.

<table>
<thead>
<tr>
<th>Females</th>
<th>Males</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>success</strong> --- p1</td>
<td><strong>success</strong> --- p1</td>
</tr>
<tr>
<td>success --- p2</td>
<td>success --- p2</td>
</tr>
<tr>
<td>p1 --- personable</td>
<td>p2 --- personable</td>
</tr>
<tr>
<td>q1 --- quality</td>
<td>q2 --- quality</td>
</tr>
<tr>
<td>p2 --- personable</td>
<td>P2 --- personable</td>
</tr>
<tr>
<td>q2 --- quality</td>
<td>q1 --- quality</td>
</tr>
<tr>
<td>s1 --- success</td>
<td>s1 --- success</td>
</tr>
<tr>
<td>s2 --- success</td>
<td>s2 --- success</td>
</tr>
<tr>
<td>s3 --- success</td>
<td>s3 --- success</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>.746</td>
<td>.523</td>
</tr>
<tr>
<td>.391</td>
<td>.548</td>
</tr>
<tr>
<td>.868</td>
<td>.841</td>
</tr>
<tr>
<td>.816</td>
<td>.863</td>
</tr>
<tr>
<td>.744</td>
<td>.845</td>
</tr>
<tr>
<td>.896</td>
<td>.856</td>
</tr>
<tr>
<td>.973</td>
<td>.940</td>
</tr>
<tr>
<td>.970</td>
<td>.929</td>
</tr>
<tr>
<td>.974</td>
<td>.972</td>
</tr>
</tbody>
</table>
Question 1: Measurement Models

<table>
<thead>
<tr>
<th>Model</th>
<th>NPAR</th>
<th>CMIN</th>
<th>DF</th>
<th>P</th>
<th>CMIN/DF</th>
<th>P value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Default model</td>
<td>34</td>
<td>15.257</td>
<td>22</td>
<td>.851</td>
<td>.693</td>
<td>.851</td>
</tr>
<tr>
<td>Model Number 2</td>
<td>27</td>
<td>16.792</td>
<td>29</td>
<td>.965</td>
<td>.579</td>
<td>.579</td>
</tr>
</tbody>
</table>

Likelihood Ratio Test (for nested models):
Default: if AIC = 83.26, and s=34, then –2LL=83.26-68=15.26
#2: if AIC=70.79, and s=27, then –2LL=70.79-54=16.79

Difference = 1.53~χ² with 7 d.f.
So, constrained model is not a significantly worse fit.
Caveat: this is an asymptotic result which needs a large N.

Question 2: Structural Model

OK, so assuming the measurement model is the same across Males and females, now we can ask if the relationship between Personableness and success is the same across gender.

Fit two models (both with equal measurement models)
1) Where ponsfem and ponsmale are both estimated (already done, this was model # 2
2) Where ponsfem and ponsmale are forced to be equal (model 3)
Constraining a Weight

![Image of software interface](image1.png)

**Question 2: Structural Model**

<table>
<thead>
<tr>
<th>Regression Weights: (grp fem - Model Number 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimate</td>
</tr>
<tr>
<td>----------</td>
</tr>
<tr>
<td>success</td>
</tr>
<tr>
<td>success</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Regression Weights: (grp male - Model Number 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-----------------------------------------------</td>
</tr>
<tr>
<td>Estimate</td>
</tr>
<tr>
<td>----------</td>
</tr>
<tr>
<td>success</td>
</tr>
<tr>
<td>success</td>
</tr>
</tbody>
</table>
Question 2: Structural Model

Likelihood Ratio Test (for nested models):

#2: if AIC=70.79, and s=27, then \(-2\text{LL}=70.79-54=16.79\)

#3: if AIC=93.67, and s=26, then \(-2\text{LL}=93.67-52=41.67\)

Difference = 24.88~\(\chi^2\) with 1 d.f.
So, constrained model is significantly worse fit.
Caveat: this is an asymptotic result which needs a large N.

Multiple group comparison for more than two groups

Compare previous model across African-American, Hispanic, and white respondents

Default model: compare measurement model to saturated model

Model 2: Constrain parameter of interest to be the same across all groups

Results:

<table>
<thead>
<tr>
<th>Model</th>
<th>NPAR</th>
<th>CMIN</th>
<th>DF</th>
<th>P</th>
<th>CMIN/DF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Default model</td>
<td>37</td>
<td>54.217</td>
<td>47</td>
<td>0.218</td>
<td>1.154</td>
</tr>
<tr>
<td>Model Number 2</td>
<td>35</td>
<td>96.085</td>
<td>49</td>
<td>0.000</td>
<td>1.961</td>
</tr>
<tr>
<td>Saturated model</td>
<td>84</td>
<td>0.000</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Independence model</td>
<td>21</td>
<td>1966.186</td>
<td>63</td>
<td>0.000</td>
<td>31.209</td>
</tr>
</tbody>
</table>
Three-way interaction across two groups:

For example, is difference in parameter different across male and female democrats in comparison to male and female republicans?

Procedure: Constrain male/female difference in parameters to be equal across political groups, and see if difference is significantly different when this constraint is not included.

Three-way interaction across more than two groups:

For example, is difference in parameter different across male and females different for republicans, democrats, and/or independents?

Procedure: Constrain male/female difference in parameters to be equal across all political groups, and see if difference is significantly different when this constraint is not included.