Factor Analysis

September 27 and October 2, 2006

Statistics for Psychosocial Research

Lectures 8 and 9

Motivating Example: Frailty

- We have a concept of what “frailty” is, but we can’t measure it directly.
- We think it combines strength, weight, speed, agility, balance, and perhaps other “factors”
- We would like to be able to describe the components of frailty with a summary of strength, weight, etc.

Factor Analysis

- Data reduction tool
- Removes redundancy or duplication from a set of correlated variables
- Represents correlated variables with a smaller set of “derived” variables.
- Factors are formed that are relatively independent of one another.
- Two types of “variables”:
  - latent variables: factors
  - observed variables

Frailty Variables

Speed of fast walk (+)
Speed of usual walk (+)
Time to do chair stands (-)
Arm circumference (+)
Body mass index (+)
Tricep skinfold thickness (+)
Shoulder rotation (+)
Upper extremity strength (+)
Pinch strength (+)
Grip strength (+)
Knee extension (+)
Hip extension (+)
Time to do Pegboard test (-)

Other examples

- Diet
- Air pollution
- Personality
- Customer satisfaction
- Depression

Applications of Factor Analysis

1. Identification of Underlying Factors:
   - clusters variables into homogeneous sets
   - creates new variables (i.e. factors)
   - allows us to gain insight to categories
2. Screening of Variables:
   - identifies groupings to allow us to select one variable to represent many
   - useful in regression (recall collinearity)

Applications of Factor Analysis

3. Summary:
   - Allows us to describe many variables using a few factors
4. Sampling of variables:
   - helps select small group of variables of representative variables from larger set
5. Clustering of objects:
   - Helps us to put objects (people) into categories depending on their factor scores

"Perhaps the most widely used (and misused) multivariate [technique] is factor analysis. Few statisticians are neutral about this technique. Proponents feel that factor analysis is the greatest invention since the double bed, while its detractors feel it is a useless procedure that can be used to support nearly any desired interpretation of the data. **The truth, as is usually the case, lies somewhere in between**. Used properly, factor analysis can yield much useful information; when applied blindly, without regard for its limitations, it is about as useful and informative as Tarot cards. **In particular, factor analysis can be used to explore the data for patterns, confirm our hypotheses, or reduce the Many variables to a more manageable number.**

-- Norman Streiner, *PDQ Statistics*

Orthogonal One Factor Model

Classical Test Theory Idea:

Ideal:

$$X_1 = F + e_1 \qquad \text{var}(e_j) = \text{var}(e_k),\ j \neq k$$
$$X_2 = F + e_2$$
$$\vdots$$
$$X_m = F + e_m$$

Reality:

$$X_1 = \lambda_1 F + e_1 \qquad \text{var}(e_j) \neq \text{var}(e_k),\ j \neq k$$
$$X_2 = \lambda_2 F + e_2$$
$$\vdots$$
$$X_m = \lambda_m F + e_m$$

(unequal “sensitivity” to change in factor)
(Related to Item Response Theory (IRT))

Key Concepts

- F is latent (i.e.unobserved, underlying) variable
- X's are observed (i.e. manifest) variables
- e_j is measurement error for X_j.
- λ_j is the "loading" for X_j.

Assumptions of Factor Analysis Model

- Measurement error has constant variance and is, on average, 0.

  $Var(e_j) = \sigma_j^2 \qquad E(e_j) = 0$

- No association between the factor and measurement error

  $Cov(F, e_j) = 0$

- No association between errors:

  $Cov(e_j, e_k) = 0$

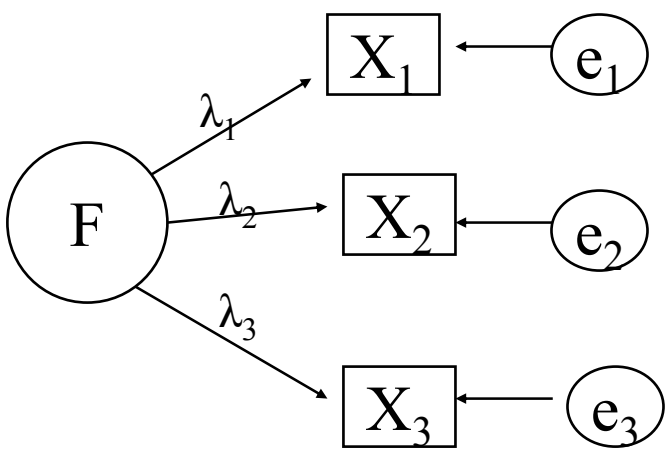
- **Local (i.e. conditional) independence: Given the factor, observed variables are independent of one another.**

  $\mathbf{Cov(X_j, X_k \mid F) = 0}$

Brief Aside in Path Analysis

Local (i.e. conditional) independence: Given the factor, observed variables are independent of one another.

$$Cov(X_j , X_k \mid F) = 0$$



X's are only related to each other through their common relationship with F.

Optional Assumptions

- We will make these to simplify our discussions
- F is “standardized” (think “standard normal”)

  $Var(F) = 1 \qquad E(F) = 0$

- X’s are standardized:
  - In practice, this means that we will deal with “correlations” versus “covariance”
  - This “automatically” happens when we use correlation in factor analysis, so it is not an extra step.

Some math associated with the **ONE FACTOR** model

- λ_j^2 is also called the "communality" of X_j in the one factor case (notation: h_j^2)
- **For standardized X_j , Corr(F, X_j) = λ_j**
- The percentage variability in (standardized) X_j explained by F is λ_j^2. (like an R^2)
- If X_j is N(0,1), then λ_j is equivalent to:
  - the slope in a regression of X_j on F
  - the correlation between F and X_j
- Interpretation of λ_j:
  - standardized regression coefficient (regression)
  - path coefficient (path analysis)
  - factor loading (factor analysis)

Some more math associated with the ONE factor model

- $\text{Corr}(X_j , X_k) = \lambda_j \lambda_k$
- Note that the correlation between X_j and X_k is completely determined by the common factor. Recall $\text{Cov}(e_j, e_k) = 0$
- Factor loadings (λ_j) are equivalent to correlation between factors and variables when only a **SINGLE** common factor is involved.

Steps in Exploratory Factor Analysis

(1) Collect and explore data: choose relevant variables.

(2) Extract initial factors (via principal components)

(3) Choose number of factors to retain

(4) Choose estimation method, estimate model

(5) Rotate and interpret

(6) (a) Decide if changes need to be made (e.g. drop item(s), include item(s))

(b) repeat (4)-(5)

(7) Construct scales and use in further analysis

Data Exploration

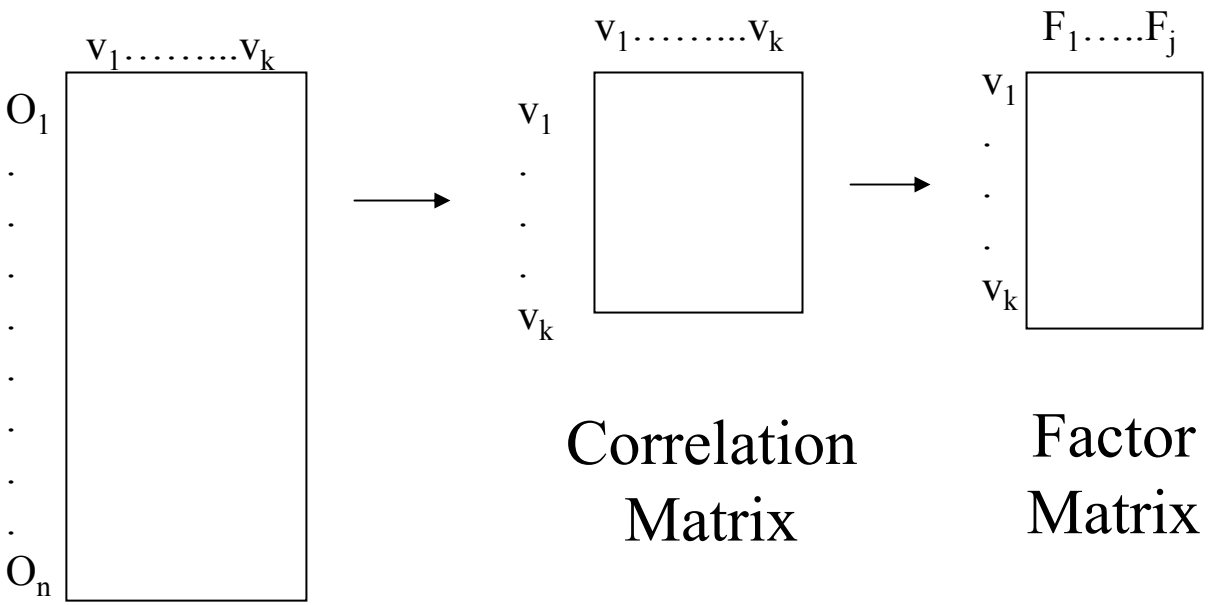
- Histograms
  - normality
  - discreteness
  - outliers
- Covariance and correlations between variables
  - very high or low correlations?
- Same scale
- high = good, low = bad?

Aside: Correlation vs. Covariance

- >90% of Factor Analyses use correlation matrix
- <10% use covariance matrix
- We will focus on correlation matrix because
  - It is less confusing than switching between the two
  - It is much more commonly used and more commonly applicable
- Covariance does have its place (we'll address that next time).

Data Matrix

- Factor analysis is **totally dependent** on correlations between variables.
- Factor analysis summarizes correlation structure



Implications for assumptions about X's?

Frailty Example
(N=571)

	arm	ski	fastw	grip	pincr	upex	knee	hipext	shldr	peg	bmi	usalk
skinfld	**0.71**											
fastwalk	-0.01	0.13										
gripstr	**0.34**	0.26	0.18									
pinchstr	**0.34**	**0.33**	0.16	**0.62**								
upextstr	0.12	0.14	0.26	**0.31**	0.25							
kneeext	0.16	**0.31**	**0.35**	0.28	0.28	0.21						
hipext	0.11	0.28	0.18	0.24	0.24	0.15	**0.56**					
shldrrot	0.03	0.11	0.25	0.18	0.19	**0.36**	**0.30**	0.17				
pegbrd	-0.10	-0.17	**-0.34**	-0.26	-0.13	-0.21	-0.15	-0.11	-0.15			
bmi	**0.88**	**0.64**	-0.09	0.25	0.28	0.08	0.13	0.13	0.01	-0.04		
uslwalk	-0.03	0.09	**0.89**	0.16	0.13	0.27	**0.30**	0.14	0.22	**-0.31**	-0.10	
chrstand	0.01	-0.09	-0.43	-0.12	-0.12	-0.22	-0.27	-0.15	-0.09	0.25	0.03	**-0.42**

One Factor Model

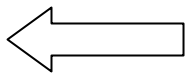
$$X_1 = \lambda_1 F + e_1$$
$$X_2 = \lambda_2 F + e_2$$

...

$$X_m = \lambda_m F + e_m$$

One Factor Frailty Solution

Variable	Loadings
arm_circ	0.28
skinfld	0.32
fastwalk	0.30
gripstr	0.32
pinchstr	0.31
upextstr	0.26
kneeext	0.33
hipext	0.26
shldrrot	0.21
pegbrd	-0.23
bmi	0.24
uslwalk	0.28
chrstand	-0.22



These numbers represent the correlations between the common factor, F, and the input variables.

Clearly, estimating **F** is **part** of the process

More than One Factor

- m factor orthogonal model
- ORTHOGONAL = INDEPENDENT
- Example: frailty has domains, including strength, flexibility, speed.
- m factors, n observed variables

$$X_1 = \lambda_{11}F_1 + \lambda_{12}F_2 + \ldots + \lambda_{1m}F_m + e_1$$
$$X_2 = \lambda_{21}F_1 + \lambda_{22}F_2 + \ldots + \lambda_{2m}F_m + e_2$$
$$\ldots\ldots$$
$$X_n = \lambda_{n1}F_1 + \lambda_{n2}F_2 + \ldots + \lambda_{nm}F_m + e_n$$

More than One Factor

- Matrix notation: $\mathbf{X}_{nx1} = \mathbf{\Lambda}_{nxm}\mathbf{F}_{mx1} + \mathbf{e}_{nx1}$
- Same general assumptions as one factor model.
  - $corr(F_s, x_j) = \lambda_{js}$
- Plus:
  - $corr(F_s, F_r) = 0$ for all $s \neq r$ (i.e. orthogonal)
  - this is **forced** independence
  - simplifies covariance structure
  - $corr(x_i, x_j) = \lambda_{i1}\lambda_{j1} + \lambda_{i2}\lambda_{j2} + \lambda_{i3}\lambda_{j3} + \ldots$
- To see details of dependent factors, see Kim and Mueller.

Matrix notation:

$$\mathbf{X_{nx1} = \Lambda_{nxm} F_{mx1} + e_{nx1}}$$

$$\begin{bmatrix} X_1 \\ \vdots \\ \vdots \\ X_n \end{bmatrix}_{nx1} = \begin{bmatrix} \lambda_{11} & \cdots & \cdots & \lambda_{1m} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \lambda_{n1} & \cdots & \cdots & \lambda_{nm} \end{bmatrix}_{nxm} \begin{bmatrix} F_1 \\ \vdots \\ F_m \end{bmatrix}_{mx1} + \begin{bmatrix} e_1 \\ \vdots \\ \vdots \\ e_n \end{bmatrix}_{nx1}$$

Factor Matrix

$$\begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1m} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n1} & \cdots & \lambda_{nm} \end{bmatrix}_{nxm}$$

- Columns represent derived factors
- Rows represent input variables
- Loadings represent degree to which each of the variables “correlates” with each of the factors
- Loadings range from -1 to 1
- Inspection of factor loadings reveals extent to which each of the variables contributes to the meaning of each of the factors.
- High loadings provide meaning and interpretation of factors (~ regression coefficients)

Frailty Example

	Factors				
Variable	size	speed	Hand strength	Leg strength	Uniqueness
arm_circ	**0.97**	-0.01	0.16	0.01	**0.02**
skinfld	**0.71**	0.10	0.09	0.26	0.40
fastwalk	-0.01	**0.94**	0.08	0.12	0.08
gripstr	0.19	0.10	**0.93**	0.10	0.07
pinchstr	0.26	0.09	**0.57**	0.19	0.54
upextstr	0.08	0.25	0.27	0.14	**0.82**
kneeext	0.13	0.26	0.16	**0.72**	0.35
hipext	0.09	0.09	0.14	**0.68**	0.48
shldrrot	0.01	0.22	0.14	0.26	0.85
pegbrd	-0.07	-0.33	-0.22	-0.06	0.83
bmi	**0.89**	-0.09	0.09	0.04	0.18
uslwalk	-0.03	**0.92**	0.07	0.07	0.12
chrstand	0.02	-0.43	-0.07	-0.18	0.77

Communalities

- The communality of X_j is the <u>proportion of the variance of X_j explained by the *m* common factors</u>:

$$Comm(X_j) = \sum_{i=1}^{m} \lambda_{ij}^2$$

- Recall one factor model: What was the interpretation of λ_j^2?

$$Comm(X_j) = \lambda_j^2$$

- In other words, it can be thought of as the sum of squared multiple-correlation coefficients between the X_j and the factors.

- ***Uniqueness(X_j) = 1 - Comm(X_j)***

Communality of X_j

- "Common" part of variance
  - covariance between X_j and the part of X_j due to the underlying factors
  - For standardized X_j:
    - 1 = communality + uniqueness
    - uniqueness = 1 – communality
  - Can think of uniqueness = var(e_j)

➔ If X_j is informative, communality is high
➔ If X_j is <u>not</u> informative, uniqueness is high

- <u>Intuitively</u>: variables with high communality share more in common with the rest of the variables.

Communalities

- *Unstandardized X's:*
  - *Var(X) = Var(F) + Var(e)*
  - *Var(X) = Communality + Uniqueness*
  - *Communality* $\approx$ *Var(F)*
  - *Uniqueness* $\approx$ *Var(e)*

- *How can Var(X)=Var(F)= 1 when using standardized variables? That implies that Var(e)=0.*
  - *After Var(F) is derived, then F is 'standardized' to have variance of 1. Two step procedure.*
  - Actual variances are "irrelevant" when using correlations and/or standardized X's.

How many factors?

- Intuitively: The number of uncorrelated constructs that are jointly measured by the X's.
- Only useful if number of factors is less than number of X's (recall "data reduction").
- Identifiability: Is there enough information in the data to estimate all of the parameters in the factor analysis? May be constrained to a certain number of factors.

Choosing Number of Factors

Use “principal components” to help decide

- type of factor analysis
- number of factors is equivalent to number of variables
- each factor is a weighted combination of the input variables:

$$F_1 = a_{11}X_1 + a_{12}X_2 + \ldots.$$

- Recall: For a factor analysis, <u>generally</u>,

$$X_1 = a_{11}F_1 + a_{12}F_2 + \ldots$$

Estimating Principal Components

- **The first PC is the linear combination with maximum variance**
- That is, it finds vector a_1 to maximize

$$Var(F_1) = Var(a_1^T X) = a_1^T Cov(X) a_1$$

- (Can use correlation instead, equation is more complicated looking)
- Constrained such that $\Sigma a_1^2 = 1$
- First PC: linear combination a_1X that maximizes $Var(a_1^T X)$ such that $\Sigma a_1^2 = 1$
- Second PC: linear combination a_2X that maximizes $Var(a_2^T X)$ such that $\Sigma a_2^2 = 1$ AND $Corr(a_1^T X, a_2^T X)=0$.
- And so on…..

Eigenvalues

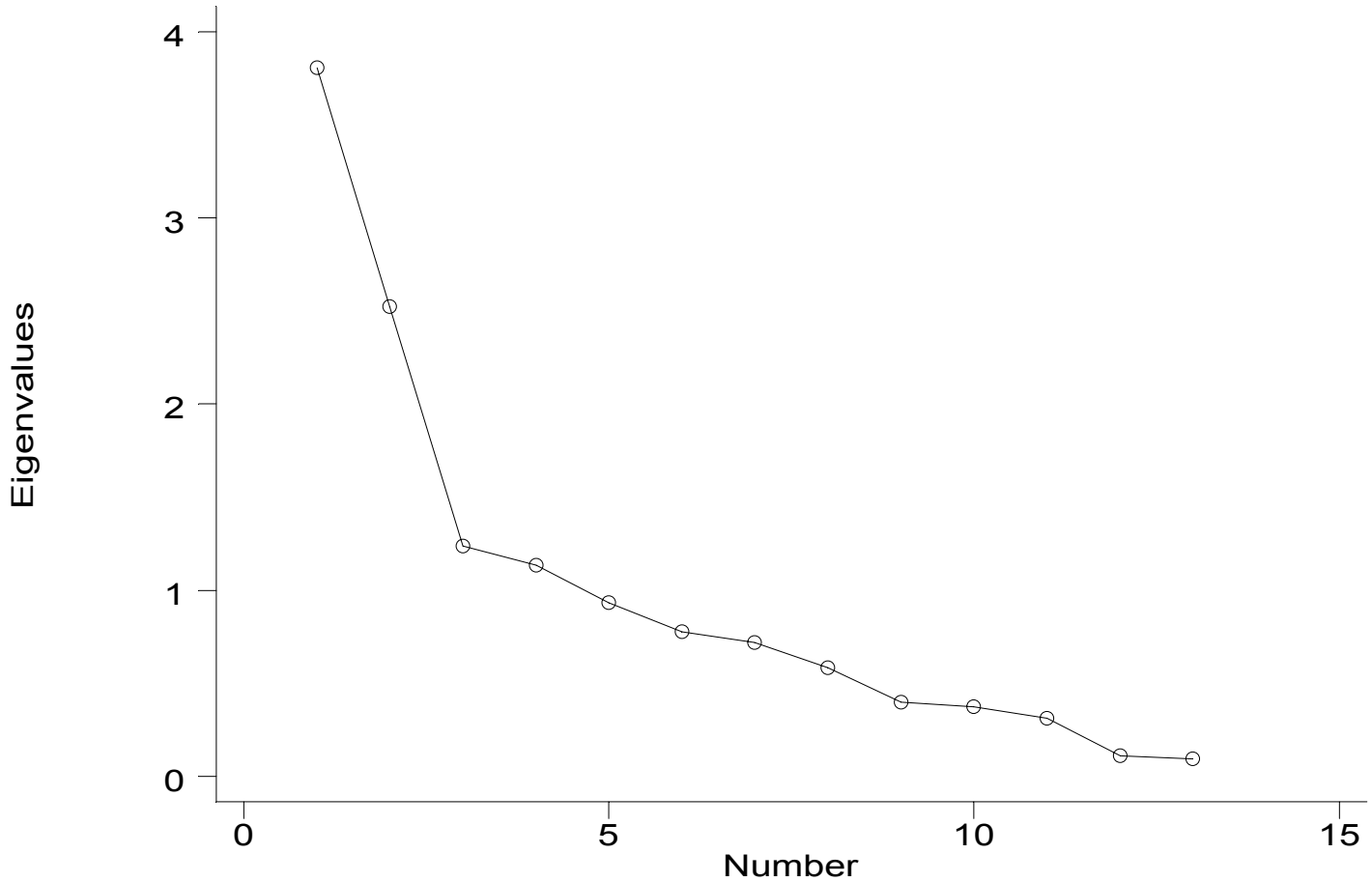
- To select how many factors to use, consider eigenvalues from a principal components analysis
- Two interpretations:
  - eigenvalue ≅ equivalent number of variables which the factor represents
  - eigenvalue ≅ amount of variance in the data described by the factor.
- Rules to go by:
  - number of eigenvalues > 1
  - scree plot
  - % variance explained
  - comprehensibility

Frailty Example

(principal components; 13 components retained)

Component	Eigenvalue	Difference	Proportion	Cumulative
1	3.80792	1.28489	0.2929	0.2929
2	2.52303	1.28633	0.1941	0.4870
3	1.23669	0.10300	0.0951	0.5821
4	1.13370	0.19964	0.0872	0.6693
5	0.93406	0.15572	0.0719	0.7412
6	0.77834	0.05959	0.0599	0.8011
7	0.71875	0.13765	0.0553	0.8563
8	0.58110	0.18244	0.0447	0.9010
9	0.39866	0.02716	0.0307	0.9317
10	0.37149	0.06131	0.0286	0.9603
11	0.31018	0.19962	0.0239	0.9841
12	0.11056	0.01504	0.0085	0.9927
13	0.09552	.	0.0073	1.0000

Scree Plot for Frailty Example



First 6 factors in principal components

	Eigenvectors					
Variable	1	2	3	4	5	6
arm_circ	0.28486	0.44788	-0.26770	-0.00884	0.11395	0.06012
skinfld	0.32495	0.31889	-0.20402	0.19147	0.13642	-0.03465
fastwalk	0.29734	-0.39078	-0.30053	0.05651	0.01173	0.26724
gripstr	0.32295	0.08761	0.24818	-0.37992	-0.41679	0.05057
pinchstr	0.31598	0.12799	0.27284	-0.29200	-0.38819	0.27536
upextstr	0.25737	-0.11702	0.17057	-0.38920	0.37099	-0.03115
kneeext	0.32585	-0.09121	0.30073	0.45229	0.00941	-0.02102
hipext	0.26007	-0.01740	0.39827	0.52709	-0.11473	-0.20850
shldrrot	0.21372	-0.14109	0.33434	-0.16968	0.65061	-0.01115
pegbrd	-0.22909	0.15047	0.22396	0.23034	0.11674	0.84094
bmi	0.24306	0.47156	-0.24395	0.04826	0.14009	0.02907
uslwalk	0.27617	-0.40093	-0.32341	0.02945	0.01188	0.29727
chrstand	-0.21713	0.27013	0.23698	-0.10748	0.19050	0.06312

At this stage….

- Don’t worry about interpretation of factors!
- Main concern: whether a smaller number of factors can account for variability
- Researcher (i.e. YOU) needs to:
  - provide number of common factors to be extracted OR
  - provide objective criterion for choosing number of factors (e.g. scree plot, % variability, etc.)

Rotation

- In principal components, the first factor describes most of variability.
- After choosing number of factors to retain, we want to spread variability more evenly among factors.
- To do this we “rotate” factors:
  - redefine factors such that loadings on various factors tend to be very high (-1 or 1) or very low (0)
  - intuitively, it makes sharper distinctions in the meanings of the factors
  - **We use “factor analysis” for rotation NOT principal components!**

Aside

Principal factors vs. principal components. The defining characteristic that distinguishes between the two factor analytic models is that in principal components analysis we assume that *all* variability in an item should be used in the analysis, while in principal factors analysis we only use the variability in an item that it has in common with the other items. In most cases, these two methods usually yield very similar results. However, principal components analysis is often preferred as a method for data reduction, while **principal factors analysis is often preferred when the goal of the analysis is to detect structure.**

(http://www.statsoft.com/textbook/stfacan.html)

5 Factors, **Unrotated**

	Factor Loadings					
Variable	1	2	3	4	5	Uniqueness
arm_circ	**0.59934**	**0.67427**	-0.26580	-0.04146	0.02383	**0.11321**
skinfld	**0.62122**	**0.41768**	-0.13568	0.16493	0.01069	0.39391
fastwalk	**0.57983**	**-0.64697**	**-0.30834**	-0.00134	-0.05584	**0.14705**
gripstr	**0.57362**	0.08508	**0.31497**	**-0.33229**	-0.13918	0.43473
pinchstr	**0.55884**	0.13477	**0.30612**	-0.25698	-0.15520	0.48570
upextstr	**0.41860**	-0.15413	0.14411	-0.17610	0.26851	0.67714
kneeext	**0.56905**	-0.14977	0.26877	**0.36304**	-0.01108	0.44959
hipext	**0.44167**	-0.04549	**0.31590**	**0.37823**	-0.07072	0.55500
shldrrot	**0.34102**	-0.17981	0.19285	-0.02008	**0.31486**	0.71464
pegbrd	**-0.37068**	0.19063	0.04339	0.12546	-0.03857	0.80715
bmi	**0.51172**	**0.70802**	-0.24579	0.03593	0.04290	**0.17330**
uslwalk	**0.53682**	**-0.65795**	**-0.33565**	-0.03688	-0.05196	**0.16220**
chrstand	**-0.35387**	**0.33874**	0.07315	-0.03452	0.03548	0.75223

5 Factors, Rotated

(varimax rotation)

	Rotated Factor Loadings					
Variable	1	2	3	4	5	Uniqueness
arm_circ	-0.00702	**0.93063**	0.14300	0.00212	0.01487	**0.11321**
skinfld	0.11289	**0.71998**	0.09319	0.25655	0.02183	0.39391
fastwalk	**0.91214**	-0.01357	0.07068	0.11794	0.04312	**0.14705**
gripstr	0.13683	0.24745	**0.67895**	0.13331	0.08110	0.43473
pinchstr	0.09672	0.28091	**0.62678**	0.17672	0.04419	0.48570
upextstr	0.25803	0.08340	0.28257	0.10024	**0.39928**	0.67714
kneeext	0.27842	0.13825	0.16664	**0.64575**	0.09499	0.44959
hipext	0.11823	0.11857	0.15140	**0.62756**	0.01438	0.55500
shldrrot	0.20012	0.01241	0.16392	0.21342	**0.41562**	0.71464
pegbrd	**-0.35849**	-0.09024	-0.19444	-0.03842	-0.13004	0.80715
bmi	-0.09260	**0.90163**	0.06343	0.03358	0.00567	**0.17330**
uslwalk	**0.90977**	-0.03758	0.05757	0.06106	0.04081	**0.16220**
chrstand	**-0.46335**	0.01015	-0.08856	-0.15399	-0.03762	0.75223

2 Factors, Unrotated

Variable	Factor Loadings 1	2	Uniqueness
arm_circ	0.62007	0.66839	0.16876
skinfld	0.63571	0.40640	0.43071
fastwalk	0.56131	-0.64152	0.27339
gripstr	0.55227	0.06116	0.69126
pinchstr	0.54376	0.11056	0.69210
upextstr	0.41508	-0.16690	0.79985
kneeext	0.55123	-0.16068	0.67032
hipext	0.42076	-0.05615	0.81981
shldrrot	0.33427	-0.18772	0.85303
pegbrd	-0.37040	0.20234	0.82187
bmi	0.52567	0.69239	0.24427
uslwalk	0.51204	-0.63845	0.33020
chrstand	-0.35278	0.35290	0.75101

2 Factors, Rotated

(varimax rotation)

Rotated Factor Loadings

Variable	1	2	Uniqueness
arm_circ	-0.04259	**0.91073**	0.16876
skinfld	0.15533	**0.73835**	0.43071
fastwalk	**0.85101**	-0.04885	0.27339
gripstr	0.34324	**0.43695**	0.69126
pinchstr	0.30203	**0.46549**	0.69210
upextstr	**0.40988**	0.17929	0.79985
kneeext	**0.50082**	0.28081	0.67032
hipext	0.33483	0.26093	0.81981
shldrrot	**0.36813**	0.10703	0.85303
pegbrd	-0.40387	-0.12258	0.82187
bmi	-0.12585	**0.86017**	0.24427
uslwalk	**0.81431**	-0.08185	0.33020
chrstand	-**0.49897**	-0.00453	0.75101

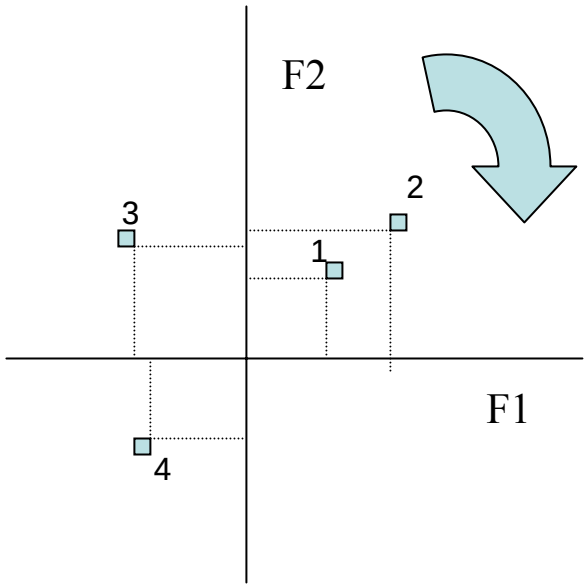
Unique Solution?

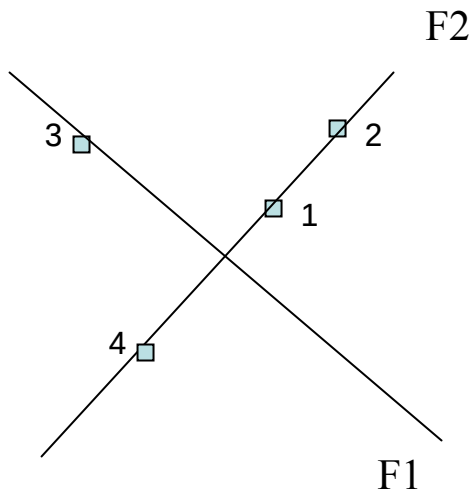
- The factor analysis solution is NOT unique!
- More than one solution will yield the same "result."
- We will understand this better by the end of the lecture…..

Rotation (continued)

- Uses “ambiguity” or non-uniqueness of solution to make interpretation simpler.
- Where does ambiguity come in?
  - Unrotated solution is based on the idea that each factor tries to maximize variance explained, conditional on previous factors
  - What if we take that away?
  - Then, there is not one “best” solution.
- All solutions are relatively the same.
- Goal is simple structure
- Most construct validation assumes simple (typically rotated) structure.
- **Rotation does NOT improve fit!**

Rotating Factors (Intuitively)





	Factor 1	Factor 2
x1	0.5	0.5
x2	0.8	0.8
x3	-0.7	0.7
x4	-0.5	-0.5

	Factor 1	Factor 2
x1	0	0.6
x2	0	0.9
x3	-0.9	0
x4	0	-0.9

Orthogonal vs. Oblique Rotation

- Orthogonal: Factors are independent
  - **varimax**: maximize squared loading variance **across variables** (sum over factors)
  - quartimax: maximize squared loading variance **across factors** (sum over variables)
  - **Intuition: from previous picture, there is a right angle between axes**
- **Note: "Uniquenesses" remain the same!**

Orthogonal vs. Oblique Rotation

- Oblique: Factors **not** independent. Change in "angle."
  - oblimin: minimize squared loading covariance between factors.
  - **promax**: simplify orthogonal rotation by making small loadings even closer to zero.
  - Target matrix: choose "simple structure" a priori. (see Kim and Mueller)
  - **Intuition: from previous picture, angle between axes is not necessarily a right angle.**
- **Note: "Uniquenesses" remain the same!**

Promax Rotation: 5 Factors

(promax rotation)

	Rotated Factor Loadings					
Variable	1	2	3	4	5	Uniqueness
arm_circ	0.01528	**0.94103**	0.05905	-0.09177	-0.00256	0.11321
skinfld	0.06938	**0.69169**	-0.03647	0.22035	-0.00552	0.39391
fastwalk	**0.93445**	-0.00370	-0.02397	0.02170	-0.02240	0.14705
gripstr	-0.01683	0.00876	**0.74753**	-0.00365	0.01291	0.43473
pinchstr	-0.04492	0.04831	**0.69161**	0.06697	-0.03207	0.48570
upextstr	0.02421	0.02409	0.10835	-0.05299	0.50653	**0.67714**
kneeext	0.06454	-0.01491	0.00733	**0.67987**	0.06323	0.44959
hipext	-0.06597	-0.04487	0.04645	**0.69804**	-0.03602	**0.55500**
shldrrot	-0.06370	-0.03314	-0.05589	0.10885	**0.54427**	**0.71464**
pegbrd	-0.29465	-0.05360	-0.13357	0.06129	-0.13064	**0.80715**
bmi	-0.07198	**0.92642**	-0.03169	-0.02784	-0.00042	0.17330
uslwalk	**0.94920**	-0.01360	-0.02596	-0.04136	-0.02118	0.16220
chrstand	**-0.43302**	0.04150	-0.02964	-0.11109	-0.00024	**0.75223**

Varimax Rotation: 5 Factors

(varimax rotation)

Variable	Rotated Factor Loadings 1	2	3	4	5	Uniqueness
arm_circ	-0.00702	**0.93063**	0.14300	0.00212	0.01487	**0.11321**
skinfld	0.11289	**0.71998**	0.09319	0.25655	0.02183	0.39391
fastwalk	**0.91214**	-0.01357	0.07068	0.11794	0.04312	**0.14705**
gripstr	0.13683	0.24745	**0.67895**	0.13331	0.08110	0.43473
pinchstr	0.09672	0.28091	**0.62678**	0.17672	0.04419	0.48570
upextstr	0.25803	0.08340	0.28257	0.10024	**0.39928**	0.67714
kneeext	0.27842	0.13825	0.16664	**0.64575**	0.09499	0.44959
hipext	0.11823	0.11857	0.15140	**0.62756**	0.01438	0.55500
shldrrot	0.20012	0.01241	0.16392	0.21342	**0.41562**	0.71464
pegbrd	**-0.35849**	-0.09024	-0.19444	-0.03842	-0.13004	0.80715
bmi	-0.09260	**0.90163**	0.06343	0.03358	0.00567	**0.17330**
uslwalk	**0.90977**	-0.03758	0.05757	0.06106	0.04081	**0.16220**
chrstand	**-0.46335**	0.01015	-0.08856	-0.15399	-0.03762	0.75223

Promax Rotation: 2 Factors

(promax rotation)

Rotated Factor Loadings

Variable	1	2	Uniqueness
arm_circ	-0.21249	**0.96331**	0.16876
skinfld	0.02708	**0.74470**	0.43071
fastwalk	**0.90259**	-0.21386	0.27339
gripstr	0.27992	**0.39268**	0.69126
pinchstr	0.23139	**0.43048**	0.69210
upextstr	**0.39736**	0.10971	0.79985
kneeext	**0.47415**	0.19880	0.67032
hipext	0.30351	0.20967	0.81981
shldrrot	**0.36683**	0.04190	0.85303
pegbrd	**-0.40149**	-0.05138	0.82187
bmi	-0.29060	**0.92620**	0.24427
uslwalk	**0.87013**	-0.24147	0.33020
chrstand	**-0.52310**	0.09060	0.75101

.

Varimax Rotation: 2 Factors

(varimax rotation)

Rotated Factor Loadings

Variable	1	2	Uniqueness
arm_circ	-0.04259	**0.91073**	0.16876
skinfld	0.15533	**0.73835**	0.43071
fastwalk	**0.85101**	-0.04885	0.27339
gripstr	0.34324	**0.43695**	0.69126
pinchstr	0.30203	**0.46549**	0.69210
upextstr	**0.40988**	0.17929	0.79985
kneeext	**0.50082**	0.28081	0.67032
hipext	0.33483	0.26093	0.81981
shldrrot	**0.36813**	0.10703	0.85303
pegbrd	-0.40387	-0.12258	0.82187
bmi	-0.12585	**0.86017**	0.24427
uslwalk	**0.81431**	-0.08185	0.33020
chrstand	-**0.49897**	-0.00453	0.75101

Which to use?

- Choice is generally not critical
- Interpretation with orthogonal is “simple” because factors are independent: Loadings are correlations.
- Structure may appear more simple in oblique, but correlation of factors can be difficult to reconcile (deal with interactions, etc.)
- Theory? Are the conceptual meanings of the factors associated?
- Oblique:
  - Loading is no longer interpretable as covariance or correlation between object and factor
  - 2 matrices: pattern matrix (loadings) and structure matrix (correlations)
- Stata: varimax, promax

Steps in Exploratory Factor Analysis

(1) Collect data: choose relevant variables.

(2) Extract initial factors (via principal components)

(3) Choose number of factors to retain

(4) Choose estimation method, estimate model

(5) Rotate and interpret

(6) (a) Decide if changes need to be made (e.g. drop item(s), include item(s))

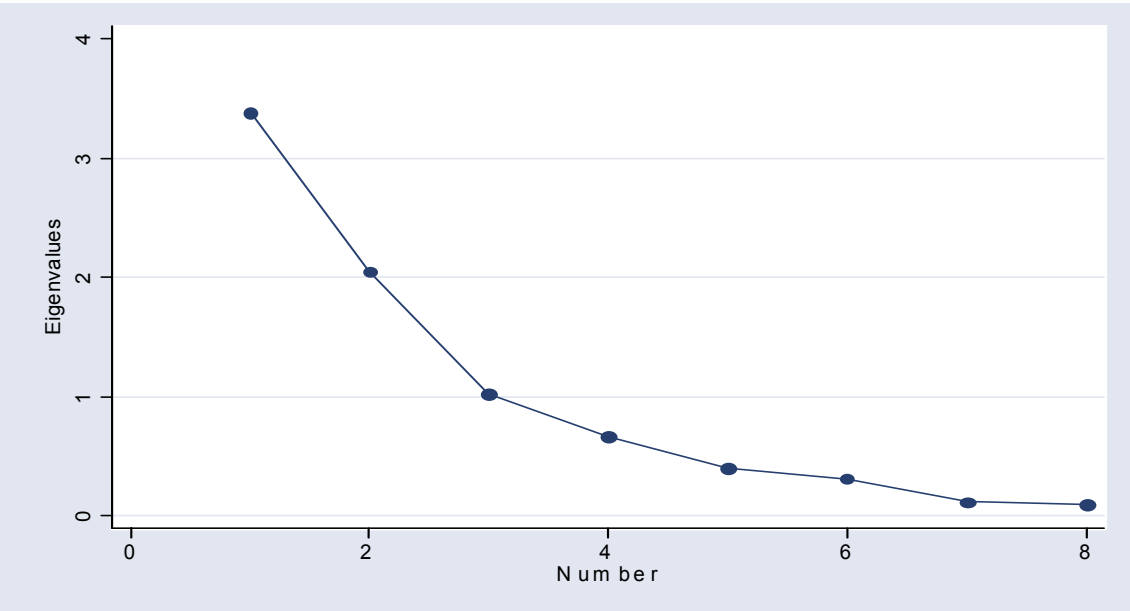
(b) repeat (4)-(5)

(7) Construct scales and use in further analysis

Drop variables with Uniqueness>0.50 in 5 factor model

```
. pca  arm_circ skinfld fastwalk gripstr pinchstr kneeext bmi uslwalk
(obs=782)

            (principal components; 8 components retained)
Component    Eigenvalue     Difference    Proportion    Cumulative
-------------------------------------------------------------------
     1        3.37554         1.32772      0.4219         0.4219
     2        2.04782         1.03338      0.2560         0.6779
     3        1.01444         0.35212      0.1268         0.8047
     4        0.66232         0.26131      0.0828         0.8875
     5        0.40101         0.09655      0.0501         0.9376
     6        0.30446         0.19361      0.0381         0.9757
     7        0.11085         0.02726      0.0139         0.9896
     8        0.08358               .      0.0104         1.0000
```



3 Factor, Varimax Rotated

(varimax rotation)
Rotated Factor Loadings

Variable	weight	Leg agility.	hand str	Uniqueness
arm_circ	**0.93225**	0.00911	-0.19238	0.09381
skinfld	**0.84253**	0.17583	-0.17748	0.22773
fastwalk	0.01214	**0.95616**	-0.11423	0.07256
gripstr	0.19156	0.13194	**-0.86476**	0.19809
pinchstr	0.20674	0.13761	**-0.85214**	0.21218
kneeext	0.22656	**0.52045**	-0.36434	0.54505
bmi	**0.92530**	-0.07678	-0.11021	0.12579
uslwalk	-0.00155	**0.95111**	-0.09161	0.08700

2 Factor, Varimax Rotated

(varimax rotation)
Rotated Factor Loadings

Variable	weight	speed	Uniqueness
arm_circ	**0.94411**	0.01522	0.10843
skinfld	**0.76461**	0.16695	0.38751
fastwalk	0.01257	**0.94691**	0.10320
gripstr	0.43430	0.33299	0.70050
pinchstr	0.44095	0.33515	0.69324
kneeext	0.29158	0.45803	0.70519
bmi	**0.85920**	-0.07678	0.25589
uslwalk	-0.00163	**0.89829**	0.19308

Uniqueness Issues

- One covariance structure can be produced by the same number of common factors with a different configuration
- One covariance structure can be produced by factor models with different numbers of common factors
- One covariance structure can be produced by a factor model and also by a non-factor analytic model.

Methods for Extracting Factors

- Principal Components (already discussed)
- Principal Factor Method
- Iterated Principal Factor / Least Squares
- Maximum Likelihood (ML)

✸ Most common(?): ML and Least Squares

Principal Factor Analysis

Uses communalities to estimate and assumes true communalities are the squared multiple correlation coefficients.

- Uniqueness(X) $\approx$ Var(e)
- Var(X) = Comm(X) + Uniqueness(X).

(1) Estimate uniqueness (i.e., var(e_j)) using $1 - R^2$ where R^2 is for a regression of X_j on all other X's.

This assumes that the "communality" of X is the same as the amount of X described by the other X's.

(2) (Correlation) Perform <u>principal components</u> on:

$$Corr(X) - Var(e) \cong Communality$$

$$\begin{pmatrix} 1 & \dots & Corr(X_1, X_n) \\ \dots & \dots & \dots \\ Corr(X_n, X_1) & \dots & 1 \end{pmatrix} - \begin{pmatrix} Var(e_1) & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & Var(e_n) \end{pmatrix} = \begin{pmatrix} 1 - Var(e_1) & \dots & Corr(X_1, X_n) \\ \dots & \dots & \dots \\ Corr(X_n, X_1) & \dots & 1 - Var(e_n) \end{pmatrix}$$

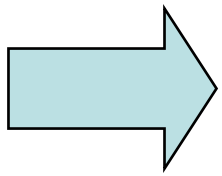
Principal Factor Analysis

- Simplified explanation
- Steps:
  1. Get initial estimates of communalities
     - squared multiple correlations
     - highest absolute correlation in row
  2. Take correlation matrix and replace diagonal elements by communalities. We call this the "adjusted" correlation matrix.
  3. Apply principal components analysis

Principal Factor Analysis

1. Obtain correlation (covariance) matrix

1	r_{12}	r_{13}	r_{14}	r_{15}	r_{16}	r_{17}
r_{21}	1	r_{23}	r_{24}	r_{25}	r_{26}	r_{27}
r_{31}	r_{32}	1	r_{34}	r_{35}	r_{36}	r_{37}
r_{41}	r_{42}	r_{43}	1	r_{45}	r_{46}	r_{47}
r_{51}	r_{52}	r_{53}	r_{55}	1	r_{56}	r_{57}
r_{61}	r_{62}	r_{63}	r_{64}	r_{65}	1	r_{67}
r_{71}	r_{72}	r_{73}	r_{74}	r_{75}	r_{76}	1



2. Replace 1's (variances) with estimate of communality

h_1^2	r_{12}	r_{13}	r_{14}	r_{15}	r_{16}	r_{17}
r_{21}	h_2^2	r_{23}	r_{24}	r_{25}	r_{26}	r_{27}
r_{31}	r_{32}	h_3^2	r_{34}	r_{35}	r_{36}	r_{37}
r_{41}	r_{42}	r_{43}	h_4^2	r_{45}	r_{46}	r_{47}
r_{51}	r_{52}	r_{53}	r_{55}	h_5^2	r_{56}	r_{57}
r_{61}	r_{62}	r_{63}	r_{64}	r_{65}	h_6^2	r_{67}
r_{71}	r_{72}	r_{73}	r_{74}	r_{75}	r_{76}	h_7^2

3. Apply principal components to "adjusted" correlation matrix and use results.

Iterative Principal Factor / Least Squares

1. Perform Principal Factor as described above.
2. Instead of stopping after principal components, reestimate communalities based on loadings.
3. Repeat until convergence

Better than without iterating!

Iterated Principal Factors / Least Squares

Standard Least Squares approach

Minimize:

$$\sum_j \left(1 - Comm(X_j) - Var(e_j)\right)^2$$

Maximum Likelihood Method

- Assume F's are normal
- Use likelihood function
- Maximize parameters
- Iterative procedure
- Notes:
  - normality matters!
  - Estimates can get "stuck" at boundary
    (e.g. communality of 0 or 1).
  - Software choice matters (e.g. SPSS vs. Stata)
  - Must rotate for interpretable solution

Choice of Method

- Give different results because they
  - use different procedures
  - use different restrictions
  - make different assumptions
- Benefit of ML
  - Can get statistics which allow you to compare factor analytic models

Which Method Should You Use?

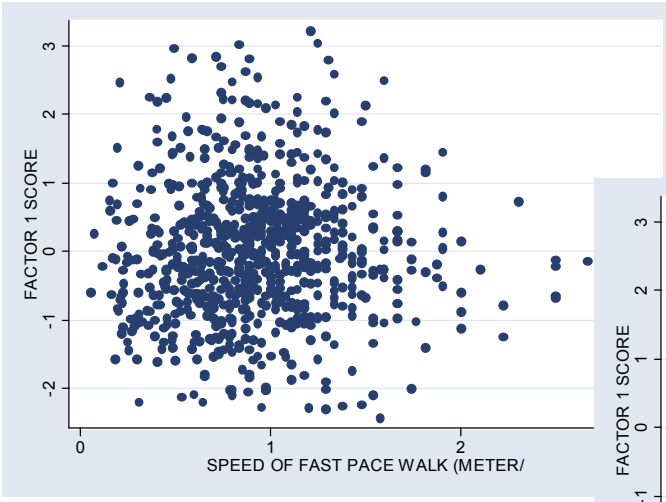
- Statisticians: PC and ML
- Social Sciences: LS and Principal Factor
- Stata:
  - **'pca' command (principal components) in Stata 8**
  - **'factor, pc' command (principal components) in Stata7**
  - 'factor, pf' (principal factor)
  - `factor, ipf' (iterated principal factor)
  - 'factor, ml' (maximum likelihood)
- ✸ Caution! *ipf* and *ml* may not converge to the right answer! Look for uniqueness of 0 or 1. Problem of "identifiability" or getting "stuck." Would be nice if it would tell you….
- ✸ For this class? I like IPF or ML, but don't always converge.

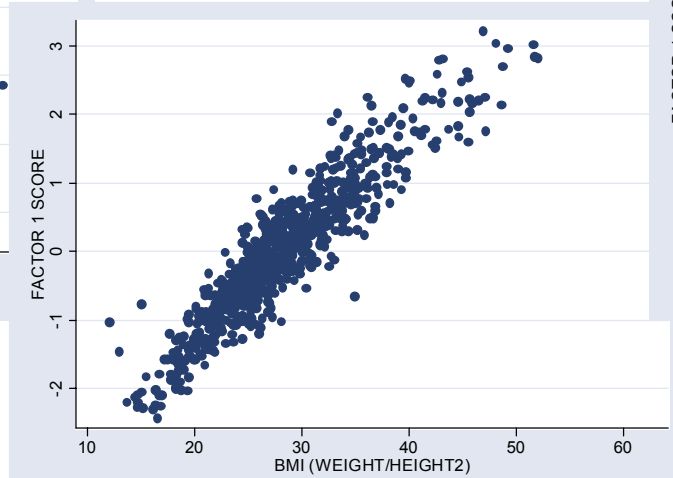
Correlation vs. Covariance?

- Correlation MUCH more commonly seen.
- If using covariance, want measures in comparable units.
- Correlation for validation, covariance for summary
  - if summary is the goal, relative variation in X's matter.
- If using covariance, do not use "number of eigenvalues > 1" or "scree plot" for determining number of factors! Nonsensical!
- Stata: all factor analyses are based on correlations. Only can use covariance in PC.

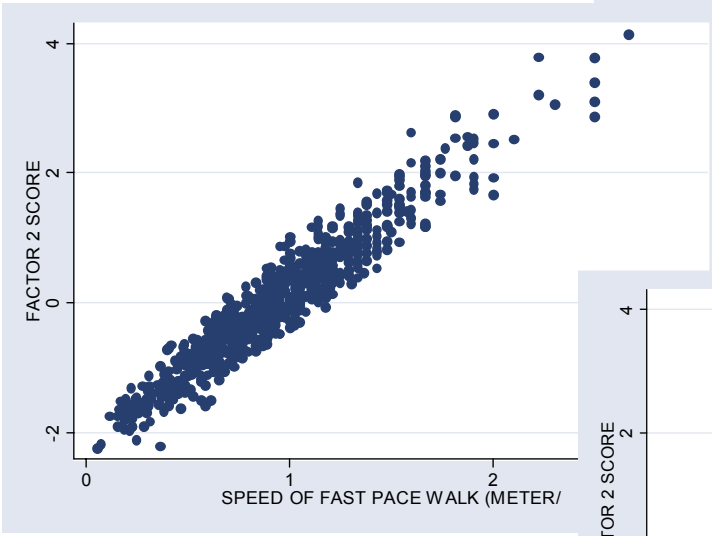
Factor Scores and Scales

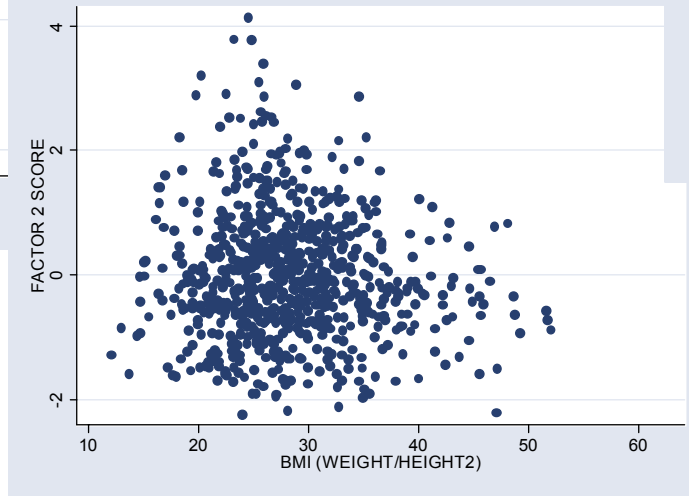
- Each object (e.g. each woman) gets a factor score for each factor.
- Old data vs. New data
- The factors themselves are variables
- “Object’s” score is weighted combination of scores on input variables
- These weights are NOT the factor loadings!
- Loadings and weights determined simultaneously so that there is no correlation between resulting factors.
- We won’t bother here with the mathematics….

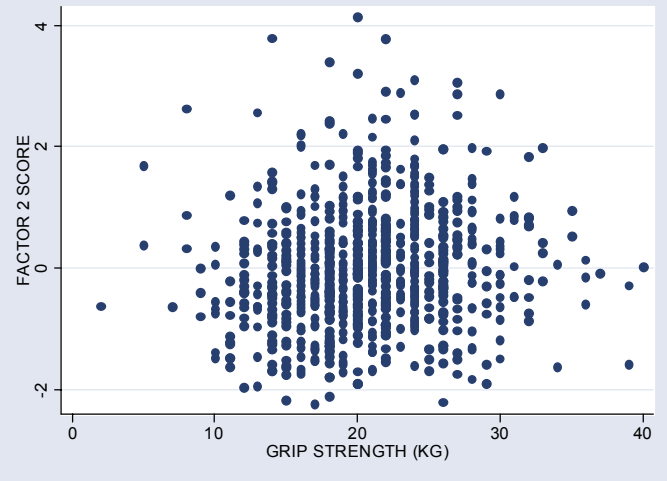

FACTOR 1 SCORE
-2
-1
0
1
2
3
0
1
2
SPEED OF FAST PACE WALK (METER/


FACTOR 1 SCORE
-2
-1
0
1
2
3
10
20
30
40
50
60
BMI (WEIGHT/HEIGHT2)


FACTOR 1 SCORE
-2
-1
0
1
2
3
0
10
20
30
40
GRIP STRENGTH (KG)


FACTOR 2 SCORE
-2
0
2
4
0
1
2
SPEED OF FAST PACE WALK (METER/


FACTOR 2 SCORE
-2
0
2
4
10
20
30
40
50
60
BMI (WEIGHT/HEIGHT2)


FACTOR 2 SCORE
-2
0
2
4
0
10
20
30
40
GRIP STRENGTH (KG)

Interpretation

- Naming of Factors
- Wrong Interpretation: Factors represent separate groups of people.
- Right Interpretation: Each factor represents a continuum along which people vary (and dimensions are orthogonal if orthogonal)

Exploratory versus Confirmatory

- Exploratory:
  - summarize data
  - describe correlation structure between variables
  - generate hypotheses
- Confirmatory (more next term)
  - testing consistency with a preconceived theory
  - A kind of structural equation modeling
  - Usually force certain loadings to zero.
  - More useful when looking for associations between factors or between factors and other observed variables.

Test Based Inference for Factor Analysis

- Likelihood ratio test:
  - Goodness of fit: compares model prediction to observed data. Sensitive to sample size.
  - Comparing models: Compare deviance statistics from different models
- Information Criteria:
  - Choose model with highest IC. Penalize for number of parameters (principle of parsimony)
  - BIC (Schwarz):

    log(L) - ln(N/2)*(number of parameters)
  - AIC (Akaike):

    log(L) - (number of parameters in model)

Factor Analysis with Categorical Observed Variables

- Factor analysis hinges on the correlation matrix
- As long as you can get an interpretable correlation matrix, you can perform factor analysis
- Binary items?
  - Tetrachoric correlation
  - Expect attenuation!
- Mixture of items?
  - Mixture of measures
  - All must be on comparable scale.

Criticisms of Factor Analysis

- Labels of factors can be arbitrary or lack scientific basis
- Derived factors often very obvious
  - defense: but we get a quantification
- “Garbage in, garbage out”
  - really a criticism of input variables
  - factor analysis reorganizes input matrix
- Too many steps that could affect results
- Too complicated
- Correlation matrix is often poor measure of association of input variables.

Stata Commands

factor y1 y2 y3,

Options:

Estimation method: *pcf, pf, ipf, and ml (default is pf)*

Number of factors to keep: *factor(#)*

Use covariance instead of correlation matrix*: cov*

* For Stata 8, to do principal components, must use '*pca*' command!

Post-factor commands:

rotation: *rotate* or *rotate, promax*

screeplot*: greigen*

generate factor scores*: score f1 f2 f3…*