Kruskal–Wallis Test

\[
H = \frac{12}{N(N+1)} \sum_{j=1}^{K} n_j \left( R_{j} - \frac{N+1}{2} \right)^2
\]

\[
= \frac{12}{N(N+1)} \sum_{j=1}^{K} \frac{R_{j}^2}{n_j} - \frac{3}{2} (N+1)
\]
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Adjustment for ties

\[ H' = \frac{H}{1 - (\text{?})} \]
K-sample permutation test

K treatments/groups

\( n_i = \# \text{ of obs. in trt. } i \)

\( \bar{Y}_i = \text{Mean of obs. for trt. } i \)

\( S_i^2 = (\text{Sample Variance}) \quad \text{for trt. } i \)

\( \bar{Y} = \text{Mean of all obs.} \)
\[ N = \sum_{i=1}^{k} n_i \left[ \sum_{ij} \frac{\hat{\theta}}{\hat{\beta}} (Y_{ij} - \bar{Y}) = 0 \right] \]

\[ TSS = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y})^2 \]

\[ SST = \sum_{i=1}^{k} n_i (\bar{Y}_i - \bar{Y})^2 \]

\[ SSE = \sum_{i=1}^{k} (n_i - 1) S_i^2 \]
\[ F_{test} = \frac{SST/(k-1)}{SSE/(N-k)} \]

\[ \sim F_{k-1, N-k} \]

We can develop a permutation based F-test.
Let the null hypothesis be

$$H_0: \ F_1(x) = F_2(x) = \ldots = F_k(x)$$

alt: $$H_A: \ F_i(x) < F_j(x) \quad \text{for at least one pair } (i, j)$$
We permute the obs. among the treatments, & for each permutation, we get a F statistic. The permutation p-value is the fraction of F's $\geq F_{\text{obs}}$; = the obs. F-stat.
Intuitive Derivation of the Chi-square approx. of KW

Under Normality

\[ \frac{SST_0}{\sigma^2} \sim \chi^2_{k-1} \]

Rank version of this is

\[ SST_R = \sum_{i=1}^{k} n_i (\bar{R_i} - \frac{N+1}{2})^2 \]

Let's assume \( E(c.(SST_R)) = k-1 \)
Now:

\[ E(\bar{R}_i - \frac{N+1}{2})^2 = \text{Var}(\bar{R}_i) \]

\[ = \left( \frac{N-h_i}{N-1} \right) \frac{\sigma^2_{R_i}}{h_i} \frac{\bar{R}_i}{\text{f.p.c.}} \left[ \frac{N-h_i}{N-1} \frac{1-h_i}{N} \right] \]

where \( \sigma^2 = \text{population variance of the combined sample.} \) \[ \rightarrow 1 \text{ as } N \to \infty \]
\[ E(SST_R) = \sum_{i=1}^{k} \left( \frac{N-n_i}{N-1} \right) \sigma_R^2 = (k-1) \frac{N \cdot \sigma_R^2}{N-1} \]
To satisfy

$$E \left[ c \cdot \text{SSTR} \right] = k-1$$

$$c = \frac{N-1}{N \sum \hat{R}^2} = \frac{N-1}{N \cdot \frac{N-1}{12}} = \frac{12}{N(N+1)}$$

$$\hat{R}^2 = \frac{N-1}{12}$$
\[ T^k = (T_1, \ldots, T_{k-1}) \]

\[ \sim \chi^{k-1} \]

\[ T_k = \text{lin. combination of } T_1, \ldots, T_{k-1} \]
Ordered alternatives

\[ U_{v'u'} = \sum_{i=1}^{n_u} \sum_{j=1}^{n_v} \phi(X_{iu}, X_{jv}) \]

\[ \phi(a, b) = \mathbb{1}, \quad a < b \]

\[ \mathbb{1}, \quad 0, \quad 0.0 \]
Sum of

\[ 1 + 2 + \ldots + (k-1) \]

\[ = \frac{(k-1)k}{2} = \binom{k}{2} \]