Supplementary material for 'Heteroscedastic nonlinear regression models based on scale mixtures of skew-normal distributions' by V. H. Lachos, D. Bandyopadhyay and A. M. Garay

First and second order derivatives in the context of the SMSN-HNLM

For the n independent responses $Y_i, i = 1, \ldots, n$, define $I_i^{\Phi}(w) = E_H[U^w e^{-Ud_i/2}\Phi_1(U^{1/2}A_i)]$ and $I_i^{\phi}(w) = \frac{1}{\sqrt{2\pi}}E_H[U^w e^{-U(d_i+A_i^2)/2}]$. So, the score function is given by $U(\boldsymbol{\theta}) = \sum_{i=1}^n U_i(\boldsymbol{\theta})$ where $U_i(\boldsymbol{\theta}) = \frac{\partial \ell_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$ and the observed information matrix $\mathbf{J}(\boldsymbol{\theta}) = -\sum_{i=1}^n \frac{\partial^2 \ell_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}^\top}$. The expressions for $U(\boldsymbol{\theta})$ and $\mathbf{J}(\boldsymbol{\theta})$ are presented below.

$$U_i(\boldsymbol{\theta}) = \frac{\partial \ell_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = -\frac{1}{2}\frac{\partial \log\sigma^2}{\partial \boldsymbol{\theta}} - \frac{1}{2}\frac{\partial \log m_i}{\partial \boldsymbol{\theta}} + \frac{1}{K_i}\frac{\partial K_i}{\partial \boldsymbol{\theta}}, \tag{1}$$

with $\frac{\partial K_i}{\partial \boldsymbol{\theta}} = I_i^{\phi}(1)\frac{\partial A_i}{\partial \boldsymbol{\theta}} - \frac{1}{2}I_i^{\Phi}(3/2)\frac{\partial d_i}{\partial \boldsymbol{\theta}}$,

and the observed information matrix is given by

$$\mathbf{J}(\boldsymbol{\theta}) = -\sum_{i=1}^n \frac{\partial^2 \ell_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}^\top}, \tag{2}$$

where

$$\frac{\partial^2 \ell_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}^\top} = -\frac{1}{2}\frac{\partial^2 \log\sigma^2}{\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}^\top} - \frac{1}{2}\frac{\partial^2 \log m_i}{\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}^\top} - \frac{1}{(K_i)^2}\frac{\partial K_i}{\partial \boldsymbol{\theta}}\frac{\partial K_i}{\partial \boldsymbol{\theta}^\top} + \frac{1}{K_i}\frac{\partial^2 K_i}{\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}^\top},$$

with

$$\begin{aligned}\frac{\partial^2 K_i}{\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}^\top} &= \frac{1}{4}I_i^{\Phi}\left(\frac{5}{2}\right)\frac{\partial d_i}{\partial \boldsymbol{\theta}}\frac{\partial d_i}{\partial \boldsymbol{\theta}^\top} - \frac{1}{2}I_i^{\Phi}\left(\frac{3}{2}\right)\frac{\partial^2 d_i}{\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}^\top} - \frac{1}{2}I_i^{\phi}(2)\left(\frac{\partial A_i}{\partial \boldsymbol{\theta}}\frac{\partial d_i}{\partial \boldsymbol{\theta}^\top} + \frac{\partial d_i}{\partial \boldsymbol{\theta}}\frac{\partial A_i}{\partial \boldsymbol{\theta}^\top}\right)\\ &\quad -I_i^{\phi}(2)A_i\frac{\partial A_i}{\partial \boldsymbol{\theta}}\frac{\partial A_i}{\partial \boldsymbol{\theta}^\top} + I_i^{\Phi}(1)\frac{\partial^2 A_i}{\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}^\top}.\end{aligned}$$

The first and second order derivatives of $d_i = B_i^2$ and $A_i = \lambda B_i$ involve standard algebraic manipulations and are obtained below, where $B_i = (y_i - \eta(\mathbf{x}_i, \boldsymbol{\beta}) - b\sigma m_i^{1/2}\delta)/(\sigma m_i^{1/2}) = C_i - b\delta$, with $C_i = (y_i - \eta(\mathbf{x}_i, \boldsymbol{\beta}))/(\sigma m_i^{1/2})$.

- d_i:

$$\begin{aligned}
\frac{\partial d_i}{\partial \boldsymbol{\beta}} &= -2\frac{B_i}{\sigma_i}\frac{\partial \eta_i}{\partial \boldsymbol{\beta}}, \quad \frac{\partial d_i}{\partial \sigma^2} = -\frac{B_i}{\sigma^2}C_i, \quad \frac{\partial d_i}{\partial \lambda} = -2bB_i\delta', \quad \frac{\partial d_i}{\partial \boldsymbol{\rho}} = -\frac{B_i}{m_i}C_i\frac{\partial m_i}{\partial \boldsymbol{\rho}}, \\
\frac{\partial^2 d_i}{\partial \boldsymbol{\beta}\partial \boldsymbol{\beta}^\top} &= 2\left[\frac{1}{\sigma_i^2}\frac{\partial \eta_i}{\partial \boldsymbol{\beta}}\frac{\partial \eta_i}{\partial \boldsymbol{\beta}^\top} + \frac{B_i}{\sigma_i}\frac{\partial^2 \eta_i}{\partial \boldsymbol{\beta}\partial \boldsymbol{\beta}^\top}\right], \\
\frac{\partial^2 d_i}{\partial \boldsymbol{\beta}\partial \sigma^2} &= \frac{1}{\sigma^2\sigma_i}\left[2B_i + b\delta\right]\frac{\partial \eta_i}{\partial \boldsymbol{\beta}}, \\
\frac{\partial^2 d_i}{\partial \boldsymbol{\beta}\partial \lambda} &= \frac{2b}{\sigma_i}\delta'\frac{\partial \eta_i}{\partial \boldsymbol{\beta}}, \\
\frac{\partial^2 d_i}{\partial \boldsymbol{\beta}\partial \boldsymbol{\rho}^\top} &= \frac{1}{\sigma_i m_i}\left[2B_i + b\delta\right]\frac{\partial \eta_i}{\partial \boldsymbol{\beta}}\frac{\partial m_i}{\partial \boldsymbol{\rho}^\top}, \\
\frac{\partial^2 d_i}{\partial \sigma^2\partial \sigma^2} &= \frac{1}{2\sigma^4}(2B_i + \frac{1}{2}b\delta)C_i, \quad \frac{\partial^2 d_i}{\partial \sigma^2\partial \lambda} = \frac{b\delta'}{\sigma^2}C_i, \quad \frac{\partial^2 B_i}{\partial \sigma^2\partial \boldsymbol{\rho}^\top} = \frac{1}{2\sigma^2 m_i}(2B_i + b\delta)C_i\frac{\partial m_i}{\partial \boldsymbol{\rho}^\top}, \\
\frac{\partial^2 d_i}{\partial \lambda\partial \lambda} &= -2b[\delta'' B_i - b(\delta')^2], \quad \frac{\partial^2 d_i}{\partial \lambda\partial \boldsymbol{\rho}} = \frac{b\delta'}{m_i}(2B_i + b\delta)\frac{\partial m_i}{\partial \boldsymbol{\rho}^\top}, \\
\frac{\partial^2 d_i}{\partial \boldsymbol{\rho}\partial \boldsymbol{\rho}^\top} &= \left[\frac{1}{2m_i^2}(4B_i + b\delta)\frac{\partial m_i}{\partial \boldsymbol{\rho}}\frac{\partial m_i}{\partial \boldsymbol{\rho}^\top} - \frac{B_i}{m_i}\frac{\partial^2 m_i}{\partial \boldsymbol{\rho}\partial \boldsymbol{\rho}^\top}\right]C_i,
\end{aligned}$$

where δ' and δ'' are the first and second order derivatives of δ with respect to λ.

- A_i:

$$\begin{aligned}
\frac{\partial A_i}{\partial \boldsymbol{\beta}} &= -\frac{\lambda}{\sigma_i}\frac{\partial \eta_i}{\partial \boldsymbol{\beta}}, \quad \frac{\partial A_i}{\partial \sigma^2} = -\frac{\lambda}{2\sigma^2}C_i, \quad \frac{\partial A_i}{\partial \lambda} = B_i - b\lambda\delta', \quad \frac{\partial A_i}{\partial \boldsymbol{\rho}} = -\frac{\lambda}{2m_i}C_i\frac{\partial m_i}{\partial \boldsymbol{\rho}}, \\
\frac{\partial^2 A_i}{\partial \boldsymbol{\beta}\partial \boldsymbol{\beta}^\top} &= -\frac{\lambda}{\sigma_i}\frac{\partial^2 \eta_i}{\partial \boldsymbol{\beta}\partial \boldsymbol{\beta}^\top}, \quad \frac{\partial^2 A_i}{\partial \boldsymbol{\beta}\partial \sigma^2} = \frac{\lambda}{\sigma^2\sigma_i}\frac{\partial \eta_i}{\partial \boldsymbol{\beta}}, \quad \frac{\partial^2 A_i}{\partial \boldsymbol{\beta}\partial \lambda} = -\frac{1}{\sigma_i}\frac{\partial \eta_i}{\partial \boldsymbol{\beta}}, \quad \frac{\partial^2 A_i}{\partial \boldsymbol{\beta}\partial \boldsymbol{\rho}^\top} = \frac{\lambda}{2m_i\sigma_i}\frac{\partial \eta_i}{\partial \boldsymbol{\beta}}\frac{\partial m_i}{\partial \boldsymbol{\rho}^\top}, \\
\frac{\partial^2 A_i}{\partial \sigma^2\partial \sigma^2} &= \frac{3\lambda}{4\sigma^4}C_i, \quad \frac{\partial^2 A_i}{\partial \sigma^2\partial \lambda} = -\frac{1}{2\sigma^2}C_i, \quad \frac{\partial^2 A_i}{\partial \sigma^2\partial \boldsymbol{\rho}^\top} = \frac{\lambda}{4m_i\sigma^2}C_i\frac{\partial m_i}{\partial \boldsymbol{\rho}^\top}, \\
\frac{\partial^2 A_i}{\partial \lambda\partial \lambda} &= -b[2\delta' + \lambda\delta''], \quad \frac{\partial^2 A_i}{\partial \lambda\partial \boldsymbol{\rho}^\top} = -\frac{1}{2m_i}C_i\frac{\partial m_i}{\partial \boldsymbol{\rho}^\top}, \\
\frac{\partial^2 A_i}{\partial \boldsymbol{\rho}\partial \boldsymbol{\rho}^\top} &= \frac{\lambda}{2}\left[\frac{3}{2m_i^2}\frac{\partial m_i}{\partial \boldsymbol{\rho}}\frac{\partial m_i}{\partial \boldsymbol{\rho}^\top} - \frac{1}{m_i}\frac{\partial^2 m_i}{\partial \boldsymbol{\rho}\partial \boldsymbol{\rho}^\top}\right]C_i.
\end{aligned}$$